A footnote on local compactness

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A standard example of an impredicative definition is that of the way-below relation [11], and thus of local compactness in (point-free) topology [6]. Local compactness can be dealt with predicatively as in [8], [5]; the particularly convenient re-formulation to be presented is based on a recent result by Peter Aczel [2], and on the observation that, in set-presented topologies, the way-below relation can be formulated predicatively (Proposition 3).

A useful simple result (Lemma 2), exploited in the predicative formulation of the full way-below relation, is also used to show that a topology isomorphic to a set-presented topology is set-presented (Theorem 5).

See [10] for facts about formal topology. Recall that for a, b in a lattice L, b is said to be *way-below* a, $b \ll a$, if for all directed subsets U of L, $a \leq \bigvee U$ implies that there is $c \in U$ such that $b \leq c$ (in more geometric terms, $b \ll a$ if any covering U of a has a finite subcovering of b).

A locale L [6] is defined to be locally compact if it is a continuous lattice in the sense of Scott [11]: for every $a \in L$, $a \leq \bigvee\{b : b \ll a\}$.

Recall the following two fundamental properties of the way-below relation:

1. $b' \leq b,\ b \ll a$ *and* $a \leq a'$ *imply* $b' \ll a'$.

   and, in continuous lattices, interpolation, i.e.

2. $b \ll a$ *implies that there is* c *with* $b \ll c \ll a$.

A second-order unrestricted quantification appears in the definition of $\ll$, so that, as it stands, it is not a ('set-level') relation in contexts such as constructive set theory [1] and Martin-Löf's type theory [7].

A predicative formulation of local compactness in formal topology is the following (cf. [8], [4], [5]; see also [12]): given any formal topology $\mathcal{S} \equiv (S, \lhd)$, for $U, V \subseteq S$, say U *way-below* V if, given any $W \subseteq S$ such that $V \lhd W$, there is a finite subset $\bar{w}$ of W such that $U \lhd \bar{w}$. We say that $\mathcal{S}$ is *locally compact* if it can be endowed with an indexed family $wb(a)(a \in S)$ of subsets of S such that

wb_1 : for all $a \in S$, $a =_{\mathcal{S}} wb(a)$,

wb_2 : for all $b \in wb(a)$, $\{b\}$ is *way-below* $\{a\}$.

(that is, the subsets $wb(x)$ have to be given in advance, since, as indicated, they cannot be defined in terms of the concept of way-below; on the other hand, the subsets $wb(x)$ have to come accompanied with a proof that, for all a, $wb(a)$ suffices to cover a, and that, for all $b \in wb(a)$, $\{b\}$ is way-below $\{a\}$)[1].

One then may check that (impredicatively) the frame *Open*(S) of formal opens of $\mathcal{S}$ is locally compact (in the sense of [6]) if and only if $\mathcal{S}$ is locally compact in the above sense.

Correspondingly to property 1., one has $U' \leq U, U$ *way-below* V *and* $V \leq V'$ *imply* U' *way-below* V'. Moreover, it is non-restrictive [4] to assume that the given family wb can always be substituted by a family wb^* such that

$$b' \lhd b,\ b \in wb^*(a) \textit{ and } \ a \lhd a' \textit{ imply } b' \in wb^*(a').$$

Concerning 2. above, we have (cf. [5])

Let $(\mathcal{S}, wb)$ be locally compact, and let U, V be subsets of S with V way-below U. Then there is a finite subset $\bar{z}$ of S such that V way-below $\bar{z}$ and $\bar{z}$ way-below U.

As (complete) regularity, compactness, etc., local compactness is invariant under isomorphisms:

Proposition 1. *$\mathcal{S}$ locally compact and $\mathcal{S} \cong \mathcal{S}'$ imply $\mathcal{S}'$ locally compact.*

Proof. Let $f : \mathcal{S} \to \mathcal{S}'$, $g : \mathcal{S}' \to \mathcal{S}$ provide the isomorphism. Define, for $a \in S'$, $wb'(a) \equiv \bigcup_{c \in g(a)} f(wb(c))$. Since, for all $a \in S'$, $a =_{\mathcal{S}'} fg(a)$ and $g(a) =_{\mathcal{S}} \bigcup_{c \in g(a)} wc(c)$, one has $a =_{\mathcal{S}'} f(\bigcup_{c \in g(a)} wc(c))$, i.e. $a =_{\mathcal{S}'} wb'(a)$ for all a. Assume now, for $a \in S'$ and $U \subseteq S'$, that $a \lhd' U$, and let $b \in wb'(a)$. Then there is $c \in g(a)$ and $d \in wb(c)$ such that $b \in f(d)$. Since $g(a) \lhd g(U)$, in particular $c \lhd g(U)$, then, by local compactness of $\mathcal{S}$, $d \lhd \{a_1, ..., a_n\} \subseteq g(U)$. There are thus $a'_1, ..., a'_n$ in U such that $a_i \in g(a'_i)$, whence $b \in f(d) \lhd' f(\{a_1, ..., a_n\}) \lhd' \{a'_1, ..., a'_n\}$, as wished.

Assume now that $\mathcal{S}$ is any set-presented topology, i.e. that there are families of subsets, $I(a)(a \in S)$, and $C(a,i)(a \in S, i \in I(a))$, $C(a,i) \subseteq S$, such that $a \lhd U \iff (\exists i \in I(a)) C(a,i) \subseteq U$ ([2], [3], [9]). In this hypothesis, using the type-theoretic principle of choice, one can show that the families I, C can be used to determine $\lhd$ as a relation between general open subsets. Notice that the choice function it yields need not respect the given equality on the domain (Bishop called these functions *operations*).

Lemma 2*. *In a set-presented topology, $U \lhd V$ if and only if there is $f \in \Pi_{x \in U} I(x)$ such that $U \lhd \bigcup_{b \in U} C(b, f(b))$ and $\bigcup_{b \in U} C(b, f(b)) \subseteq V$.*

[1]The formulation given in the context of constructive set theory in [2] is (at least conceptually) slightly more general: a topology is there said to be locally compact if for every a in S there is $U \subseteq S$ such that $a \lhd U$ and, for all b in U, b way-below a. This notion gives the above one via an application of the Strong Collection scheme [2].

Proof. Assume $U \lhd V$. Since the topology is set presented, for all $x \in U$, there is $i \in I(x)$ such that $C(x,i) \subseteq V$ and $x \lhd C(x,i)$. Thus (by the principle of choice), there is $f \in \Pi_{x \in U} I(x)$ such that $U \lhd \bigcup_{b \in U} C(b, f(b)) \subseteq V$. The converse is trivial.

Remark. Here, as some other time in the following, subsets have been loosely identified with sets, so that the given proof is quite informal; the formal version depends on which particular notion of subset one is working with. The reader may check that the presented results hold true on the base of the representation of subsets as subobject (as in Bishop's approach), or as propositional functions (as in [7]). Observe also that a corresponding version of this lemma (as well as of all the results that in the following make use of the constructive principle of choice) can be proved in constructive set theory using the presentation axiom.

The following proposition shows then that, assuming the topology to be set-presented, the way-below relation can be defined at the set level: for $U, V \subseteq S$, let

$$U \ll V \equiv (\forall f \in \Pi_{x \in V} I(x))(\exists \bar{v} \subseteq \bigcup_{b \in V} C(b, f(b)))U \lhd \bar{v},$$

with $\bar{v}$ finite.

Proposition 3*. *In any set-presented topology, U way-below V if and only if $U \ll V$.*

Proof. Assume U way-below V, and let $f \in \Pi_{x \in V} I(x)$. Then, since $V \lhd \bigcup_{b \in V} C(b, f(b))$, by U way-below V one has $\exists \bar{v} \subseteq \bigcup_{b \in V} C(b, f(b)) U \lhd \bar{v}$, with $\bar{v}$ finite, whence $U \ll V$. Conversely, let $U \ll V$ and $V \lhd W$; then, since the topology is set-presented, by Lemma 2 there is $f \in \Pi_{x \in V} I(x)$ such that $V \lhd \bigcup_{b \in V} C(b, f(b)) \subseteq W$, whence, by $U \ll V$, there is a finite $\bar{v} \subseteq \bigcup_{b \in V} C(b, f(b)) \subseteq W$; thus, U way-below V.

In particular, thus, $\ll$ is (extensionally) independent from the set-presentation.

Aczel [2] shows that in constructive set theory:

Theorem 4. *Any locally compact topology $\mathcal{S}$ is set-presented.*

The proof is choice-free, relying instead on the subset collection scheme of constructive set theory. Here is a proof (that uses the type-theoretic principle of choice) of this theorem in (standard) type theory: let $I(a)$ be the set $\{f : wb(a) \to \mathcal{P}_\omega(S) | b \lhd f(b),\ \forall b \in wb(a)\}$ and let $C(a, f)$ be $\bigcup_{b \in wb(a)} f(b)$. Assume $a \lhd U$. By wb_2, and the principle of choice, there is $f : wb(a) \to \mathcal{P}_\omega(S)$ such that, for all $b \in wb(a)$, $b \lhd f(b)$ and $f(b) \subseteq U$; thus, $(\exists f \in I(a))C(a, f) \subseteq U$. Conversely, let $(\exists f \in I(a))C(a, f) \subseteq U$. By wb_1, one has $a \lhd wb(a)$; since for all $f \in I(a)$ and $b \in wb(a)$, one has $b \lhd f(b)$, by $C(a, f) \equiv \bigcup_{b \in wb(a)} f(b) \subseteq U$ one obtains $a \lhd wb(a) \lhd \bigcup_{b \in wb(a)} f(b) \lhd U$.

This result, together with Proposition 3., permits us to rephrase conveniently, and without loss of generality, the notion of locally compact topology:

A topology $\mathcal{S}$ is a locally compact if and only if it is set-presented and $a \lhd \{b : b \ll a\}$, for all $a \in S$.

Indeed, if wb is a family of subsets of the base of a topology $\mathcal{S}$ satisfying wb_1, wb_2, then by the above theorem the topology is set-presented. Moreover, by Proposition 3., for any set-presentation $I(-), C(-,-)$, $b \in wb(a)$ implies $b \ll a$, whence $a \lhd \{b : b \ll a\}$, for all a. Vice-versa, if $\mathcal{S}$ is set-presented, and $a \lhd \{b : b \ll a\}$, for all a, the family $wb(a) = \{b : b \ll a\}$ satisfies wb_1, wb_2. (Notice that, since $\ll$ is independent from the set-presentation, a topology is locally compact in this new sense with respect to a set-presentation if it is with respect to any other).

Remark. Observe that, restricting the way-below relation to range over basic elements, Proposition 3. holds without invoking choice principles [5], so that the above re-formulation of the notion of local compactness does not require choice in the context of constructive set theory.

Observe also that, in locally compact topologies, Proposition 3. can be proved without appealing to the type-theoretic principle of choice (or the presentation axiom in constructive set theory): indeed, by the interpolation property 2. and property 1. above, one has U way-below V if and only if there is a finite subset w such that U way-below w and $w \lhd V$; then one easily checks that for arbitrary subsets U and finite subsets $w = \{x_1, ..., x_k\}$, U way-below w may be defined as

$$U \ll w \equiv (\forall i_1 \in I(x_1), ..., i_k \in I(x_k))(\exists \bar{v} \subseteq C(x_1, i_1) \cup ... \cup C(x_k, i_k))U \lhd \bar{v},$$

with $\bar{v}$ finite (and $U \ll \emptyset \iff U = \emptyset$). Thus, in locally compact topologies, one has U way-below V if and only if $(\exists w \in \mathcal{P}_{Fin}(S))\ U \ll w \ \&\ w \lhd V$[2].

It is natural to expect that properties such as local compactness or regularity are preserved under isomorphism. We end these notes showing that the property of being set-presented, too, is invariant. The constructive principle of choice is to be applied twice (the first application being hidden in the recourse to Lemma 2):

Theorem 5*. *$\mathcal{S} \cong \mathcal{S}'$ and $\mathcal{S}$ set-presented imply $\mathcal{S}'$ set-presented.*

Proof. Let $f : \mathcal{S} \to \mathcal{S}'$, $g : \mathcal{S}' \to \mathcal{S}$ provide the isomorphism, and let $\mathcal{S}$ be set-presented by $I(x)(x \in S)$, and $C(x,i)(x \in S, i \in I(x))$. Let

$$\Sigma \equiv \{\langle h, k\rangle : h \in \Pi_{x \in g(y)} I(x),\ k : [\bigcup_{c \in g(y)} C(c, h(c))] \to S'\},$$

[2] A conversation with Peter Aczel induced the author to consider this alternative.

with h, k operations in Bishop's sense. Define $I'(y)(y \in S')$, and $C'(y, \langle h, k\rangle)(y \in S', \langle h, k\rangle \in I'(y))$ by

$$I'(y) \equiv \{\langle h, k\rangle \in \Sigma : f[\bigcup_{c \in g(y)} C(c, h(c))] \lhd' Im(k)\},$$

and

$$C'(y, \langle h, k\rangle) \equiv Im(k)$$

(where $Im(k) \equiv k(\bigcup_{c \in g(y)} C(c, h(c)))$). Let now $a \lhd' U$. Then, $g(a) \lhd g(U)$, whence, by the above lemma, there is $\bar{h} \in \Pi_{x \in g(a)} I(x)$ such that

$$g(a) \lhd \bigcup_{c \in g(a)} C(c, \bar{h}(c)) \subseteq g(U).$$

Thus, $\forall d \in \bigcup_{c \in g(a)} C(c, \bar{h}(c))$ there is $b \in U$ such that $d \in g(b)$. By the principle of choice, there is then $\bar{k} : [\bigcup_{c \in g(a)} C(c, \bar{h}(c))] \to S'$ such that, for all $d \in \bigcup_{c \in g(a)} C(c, \bar{h}(c))$, $\bar{k}(d) \in U$ & $d \in g(\bar{k}(d))$. We have $\langle \bar{h}, \bar{k}\rangle \in I'(a)$, since for all $d \in \bigcup_{c \in g(a)} C(c, \bar{h}(c))$, $f(d) \subseteq fg(\bar{k}(d))$, whence $f(d) \lhd' \bar{k}(d)$ and $f(\bigcup_{c \in g(a)} C(c, \bar{h}(c))) \lhd' Im(\bar{k})$. Moreover, $C'(y, \langle \bar{h}, \bar{k}\rangle) \equiv Im(\bar{k}) \subseteq U$, since by construction, for all $d \in \bigcup_{c \in g(a)} C(c, \bar{h}(c))$, we have $\bar{k}(d) \in U$.

Conversely, assume $\langle h, k\rangle \in I'(a)$ and $C'(a, \langle h, k\rangle) \equiv Im(k) \subseteq U$. By definition, $f(\bigcup_{c \in g(a)} C(c, h(c))) \lhd' Im(k) \subseteq U$; then, since for all $c \in g(a)$, $c \lhd C(c, h(c))$, we have $g(a) \lhd \bigcup_{c \in g(a)} C(c, h(c))$, whence $a \lhd' fg(a) \lhd'$ $f(\bigcup_{c \in g(a)} C(c, h(c))) \lhd' Im(k) \subseteq U$. Thus $a \lhd' U$, as wished.

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