Inductive Definitions, Imaginary Locales, Generalized Geometric Propositional Theories

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Outline of the talk

1. Background material on CZF. Inductive definitions.

2. Ordered structures in CZF.

3. Constructive point-free topology: some history, some terminology.

4. Constructive point-free topology: frame presentations/imaginary locales.

5. Formal definitions of FrmPr/ImLoc, FT/FSp, IFT/IFSp.

6. Imaginary locales and geometric propositional logic (work in progress).

7. Limits in ImLoc.

8. Applications: Johnstone-Tychonoff embedding theorem, small FSp-homsets in $\mathrm{CZF}_{exp}+\mathrm{AC}_{\omega}$, ...

9. Concluding remarks, open problems, possible developments.

Constructive set theory: CZF

Working at the *intuitionistic generalized predicative* level it is important to be explicit on the adopted setting.

In this talk, we work in the framework of Aczel-Myhill's *constructive set theory*, CZF, and variants of it.

CZF is a subsystem of ZF (without AC); it uses intuitionistic logic in place of classical logic.

Among the modifications of the ZF axioms, two crucial ones are:

- *The Separation scheme* is weakened to the *Restricted Separation scheme*.

  $\{x \in a : \varphi(x, ...)\}$ is a set *whenever* φ *is bounded*.

  φ is *bounded* if quantifiers in φ appear in the form $\forall x \in a$, $\exists x \in b$.

- The *Powerset axiom* is weakened to the *Subset Collection scheme*.

Subset Collection is a strengthening of Myhill's *Exponentiation axiom*:

Given any two sets A, B the class B^A of functions from A to B is a set.

Constructive set theory: CZF and CZF$_{exp}$

CZF with Subset Collection replaced by Exponentiation is indicated by CZF$_{exp}$.

Intuitionistic set theory IZF = CZF + Sep + Pow.

- CZF$_{exp}$ $\subsetneqq$ CZF $\subsetneqq$ IZF $\subsetneqq$ ZF.

- CZF$_{exp}$ + EM = CZF + EM = IZF+EM = ZF.

Remark 1. The Dedekind reals R_d form a *set* in CZF, and a *proper class* in CZF$_{exp}$ (Lubarsky & Rathjen '08).

Remark 2. Although Pow(X) is not a set, it is a *class*, the class of *subsets* of X.

Class-sized objects often feature in a (generalized) predicative setting as CZF.

The class inductive definition theorem

The following important theorem of constructive set theory will be exploited in this talk.

Recall that an *inductive definition* is a class Φ of pairs (a, X).

Given an inductive definition Φ, one says that a class A is $\Phi-$*closed* if:

$$(a, X) \in \Phi, \text{ and } X \subseteq A \text{ implies } a \in A.$$

Theorem. [Class inductive definition theorem, Aczel '82]. *Given any class Φ, there is in CZF the least $\Phi-$closed class $I(\Phi)$, the class inductively defined by Φ.*

Ex. $\mathbb{N}$ is $I(\Phi)$ for $\Phi = \{(\emptyset, \emptyset)\} \cup \{(x^+, \{x\}) : x \in V\}$, $x^+ = x \cup \{x\}$.

Remark. This result holds also for CZF_{exp}, and in fact even without Exponentiation.

The set inductive definition theorem

Even when Φ is a set, $I(\Phi)$ need not be a set in CZF.

However, in CZF plus the *regular extension axiom* (REA):

Theorem [Aczel '86] (CZF+REA) *If Φ is bounded, in particular if Φ is a set, then $I(\Phi)$ is a set.*

(REA: *every set is the subset of a regular set.*)

REA is a proof-theoretically powerful axiom; REA implies that very general (transfinitely) inductively defined classes are sets.

Point-free topology in constructive set theory has (mostly) been developed over CZF+REA.

Large ordered structures

A *partially ordered class* (po-class) is a class together with a class relation on it satisfying the usual axioms for a partial order.

A *large* $\bigvee$*-semilattice* is a partially ordered class that has suprema for arbitrary *subsets*.

A *class-frame*, or *class-locale*, X is a $\bigvee$-semilattice that has a top element $\top$, binary meets, and that is such that meets distribute over suprema of arbitrary sets of elements of X.

(Aczel & Rathjen '01, Aczel, '06).

Note: considered in (I)ZF, these structures remain class-sized.

Large ordered structures (cont.)

In CZF it has no use to ask that these structures are carried by sets, as this can never be the case when they are non-degenerate (Curi '10b).

A large $\bigvee$-semilattice, or a class-frame, L is *set-generated* if it has a generating set, i.e. a subset B of L such that, for all $x \in L$,

$i.$ the class $D_x \equiv \{b \in B | b \leq x\}$ is a set,

$ii.$ $x = \bigvee D_x$

(Aczel & Rathjen '01, Aczel, '06).

In (I)ZF set-generated large $\bigvee$-semilattices are 'the same' as ordinary $\bigvee$-semilattices, and similarly for set-generated class-frames.

We shall then refer to these structures simply as $\bigvee$-*semilattices* and *frames*.

Superlarge categories of large ordered structures

A homomorphism of large $\bigvee$-semilattices is a class function that respects set-indexed joins. With these homomorphisms (set-generated large) $\bigvee$-semilattices form the category $\mathbf{JSLat}_{sg}$.

Similarly one defines the category $\mathbf{Frm}_{sg}$ of (set-generated class-) frames, and $\mathbf{PFrm}_{sg}$ of (set-generated class-) preframes.

Then, in (I)ZF,

$$\mathbf{JSLat}_{sg} \equiv \mathbf{JSLat}, \quad \mathbf{Frm}_{sg} \equiv \mathbf{Frm}, \quad \mathbf{PFrm}_{sg} \equiv \mathbf{Frm}.$$

Superlarge categories of large ordered structures (cont.)

In (I)ZF, **JSLat**, **Frm**, and **PFrm** are *locally small categories*.

In CZF, $\mathbf{JSLat}_{sg}$ and $\mathbf{PFrm}_{sg}$ fail badly to enjoy this property:

Theorem 1 *In the categories* $\mathbf{JSLat}_{sg}$ *of* $\bigvee$*-semilattices, and* $\mathbf{PFrm}_{sg}$ *of preframes, no non-singleton* $Hom(X, Y)$ *can be proved to be set-indexed in CZF.*

Proof. Ordered pointwise, $Hom(X, Y)$ has, in the two cases, the structure of $\bigvee$-semilattice and of a preframe, respectively. Via a modification of the proof that no non-trivial large $\bigvee$-semilattice or class-preframe may be carried by a set in CZF (Curi '10b), one concludes.

Remark. This result holds also with respect to systems stronger than CZF, such as CZF+REA+PA+Sep.

One may be tempted to think that the same holds for frames.

Fortunately, this is *not* the case (more about this below).

Constructive point-free topology: some history

WARNING: From now on, ‘constructive’ will stand for ‘intuitionistic and predicative’.

As we have seen, the standard notion of locale (frame, cHa) is not adequate to a constructive setting, as it *presupposes* the existence of powerobjects.

The category **FSp** of *formal spaces* (Martin-Löf-Sambin ’87, Aczel ’06) provides a presentation of the concept of locale suited for type theory, CZF:

In IZF,

$$\mathbf{FSp} \leftrightarrow \mathbf{Loc}.$$

(In CZF, **FSp** is dually equivalent to the category of set-generated class-frames).

However, some standard topological (and localic) constructions appear not to be possible for formal spaces:

given arbitrary (formal) spaces X_1, X_2, one does not even know how to define

$$X_1 \times X_2$$

in a constructive way. Similarly, no uniform method exists for constructing pullbacks, equalizers, ... (limits and colimits), method that exists, in IZF, for locales and topological spaces.

Constructive point-free topology: some history (cont.)

The full subcategory **IFSp** of **FSp**, of *inductively generated formal spaces*, has been introduced in particular to fix this problem.

Cf. (Coquand, Sambin, Smith, Valentini '03), (Aczel '06).

In IZF, again,

$$\mathbf{IFSp} \leftrightarrow \mathbf{Loc}.$$

In CZF+REA, and in (a sufficiently strong version of) CTT:

- **IFSp** has limits, in particular arbitrary products.

In CZF+sREA+DC, CTT:

- **IFSp** has colimits

(Palmgren '06, Palmgren '05, Aczel 201?, ...).

However, **IFSp** does not contain important classes of formal spaces (Curi '10a).

In particular, given any X in **(I)FSp** there is a subspace Y of X that is not in **IFSp**.

Moreover, some strong principle for the existence of inductively defined sets, as REA, is *necessary* for guaranteeing the existence of (co/)limits.

Frame presentations/Imaginary locales

In constructive set theory we shall consider an *extension* of the category of formal spaces:

the category **ImLoc** of *imaginary locales*.

In contrast with formal spaces and inductively generated formal spaces, *an imaginary locale does not have an associated (class-)frame of formal opens in CZF*. However, one still has:

in IZF,

$$\mathbf{ImLoc} \leftrightarrow \mathbf{Loc}.$$

In CZF (or CZF_{exp}), *even without REA*,

- **ImLoc** has all limits.

ImLoc can be used to obtain results concerning **FSp**, **IFSp**.

Using imaginary locales it should be possible to systematically avoid applications of strong principles for the existence of inductively defined sets, as REA, in developing point-free topology in constructive set theory.

The categories FrmPr, FT, and IFT

A *generalized covering system* on a preordered set $(S, \leq)$ is an inductive definition on S, i.e. a *class* $\Phi \subseteq S \times \text{Pow}(S)$ such that, for all (a, X) in Φ,

1. $X \subseteq \downarrow \{a\}$;

2. If $b \leq a$ there is $(b, Y) \in \Phi$ with $Y \subseteq \downarrow X$.

A *frame presentation* is a structure of the form $\mathcal{C} \equiv (S, \leq, \Phi)$, with Φ a generalized covering system on the preordered set $(S, \leq)$.

The category of frame presentations/imaginary locales

Given a frame presentation $\mathcal{C} \equiv (S, \leq, \Phi)$, consider the class $\Phi_{\leq} = \Phi \cup \{(b, \{a\}) | b \leq a\}$.

As $\Phi_{\leq}$ is an inductive definition, given any subclass U of S there exists in CZF the least *class*

$$A(U) \equiv I(\Phi_{\leq}, U)$$

inductively defined by $\Phi_{\leq}$ and containing U.

Theorem 2 *For every $a, b \in S$, and for every subclass U of S, the following holds:*

0. $\downarrow \{a\} \subseteq A(\{a\})$,

1. $U \subseteq A(U)$,

2. $U \subseteq A(V)$ *implies* $A(U) \subseteq A(V)$,

3. $A(U) \cap A(V) \subseteq A(\downarrow U \cap \downarrow V)$.

Given frame presentations $\mathcal{C}_1, \mathcal{C}_2$, we define a *homomorphism* $f : \mathcal{C}_1 \to \mathcal{C}_2$ to be a mapping

$$f : S_1 \to \mathsf{Pow}(S_2)$$

such that, for every $a, b \in S$, and for every subset U of S,

1. $S_2 \subseteq A_2(f(S_1))$,

2. $\downarrow f(a) \cap \downarrow f(b) \subseteq A_2(f(\downarrow \{a\} \cap \downarrow \{b\}))$,

3. $a \in A_1(U)$ implies $f(a) \subseteq A_2(f(U))$,

with, for $V \in \mathsf{Pow}(S_1)$, $f(V) \equiv \bigcup_{a \in V} f(a)$.

Two parallel homomorphisms $f, g : S_1 \to S_2$ are *equal* if $A_2(f(a)) = A_2(g(a))$, for all $a \in S_1$.

With this notion of morphism, frame presentations define the category **FrmPr**.

Remark 1 C. Fox, independently and for other purposes, considered in his thesis a similar (but not equivalent) category.

Formal topologies and inductively generated formal topologies

Two subcategories of this category are equivalent to two already known categories:

let **FT** be the full subcategory of **FrmPr** given by those frame presentation $\mathcal{C} \equiv (S, \leq, \Phi)$ which satisfy

(A-*smallness*) $A(U)$ *set*, for every $U \in \mathrm{Pow}(S)$,

and let **IFT** be the full subcategory of FrmPr given by those $\mathcal{C} \equiv (S, \leq, \Phi)$ *which satisfy the* A-*smallness condition*, and are such that

(Φ-*smallness*) Φ *set*,

i.e., such that Φ is an ordinary *covering system*.

FT and **IFT** are respectively (equivalent to) the categories of *formal topologies* and the category of *inductively generated formal topologies*.

Frame presentations need not present, constructively

As any $X \in \mathbf{FT}$ (in particular any $X \in \mathbf{IFT}$) satisfies the A-smallness condition, one can consider the *class*

$$Sat(X) \equiv \{A(U) : U \in Pow(S)\}$$

of *saturated subsets* of X.

Theorem 2 implies that, endowed with the operations

$$U \wedge V \equiv U \downarrow V, \quad \bigvee_{i \in I} U_i \equiv A(\bigcup_{i \in I} U_i),$$

$Sat(X)$ is a (set-generated class-)frame.

Thus, *a formal topology (i.g. or not) always defines (in fact presents) a frame*.

On the other hand, since for a general frame presentation $A(U)$ need not be a set for every U, *a frame presentation need not present a frame!*

ImLoc, FSp, IFSp

In other words, given a frame presentation, one has an associated class of 'names' for the formal opens, the class Pow(S), whereas with a formal space X one has a class of veritable 'opens', the class $Sat(X)$.

In particular, a homomorphism of frame presentations associates to a basic element of the domain only *a name* for an open of the codomain.

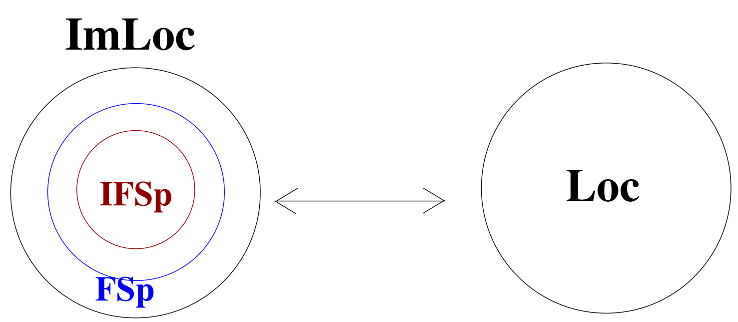
Define

- the category **ImLoc** of *imaginary locales* to be $\mathbf{FrmPr}^{op}$
- the category **FSp** of *formal spaces* to be $\mathbf{FT}^{op}$
- the category **IFSp** of *inductively generated formal spaces* to be $\mathbf{IFT}^{op}$

Adding Full Separation $\mathbf{FrmPr} = \mathbf{FT}$, and adding also Powerset, i.e. in IZF,

$$\mathbf{FrmPr} = \mathbf{FT} = \mathbf{IFT} = \mathbf{Frm},$$

thus:



Generalized geometric propositional theories, 1

Let P a set of atomic propositional symbols. A propositional geometric formula is one built from the atoms using finite conjunctions and set-indexed disjunctions.

We define a *generalized* propositional geometric theory $\mathbb{T}$ to be given by a possibly proper *class* of axioms of the form

$$\wedge\bar{p} \vdash \bigvee_{\bar{q}\in U} \wedge\bar{q},$$

for $\bar{p} \in \mathsf{Pow}_{fin}(P)$, $U \in \mathsf{Pow}(\mathsf{Pow}_{fin}(P))$. By letting

$$S_{\mathbb{T}} = \mathsf{Pow}_{fin}(P),\ \bar{p}_1 \leq_{\mathbb{T}} \bar{p}_2 \iff \bar{p}_2 \subseteq \bar{p}_1,\ \text{and } \Phi_{\mathbb{T}} = \{(\bar{l}, \downarrow U \cap \downarrow \bar{l}) : \bar{l} \leq \bar{p}\ \&\ [\wedge\bar{p} \vdash \bigvee_{\bar{q}\in U} \wedge\bar{q}] \in \mathbb{T}\},$$

one obtains a frame presentation $\mathcal{C}_{\mathbb{T}}$.

- If $\mathbb{T}$ is such that, for every $U \in \mathsf{Pow}(S_{\mathbb{T}})$, the class

  $$\{\bar{p} \in S_{\mathbb{T}} : \wedge\bar{p} \vdash \textstyle\bigvee_{\bar{q}\in U} \wedge\bar{q} \text{ is derivable in } \mathbb{T}\}\ \textit{is a set},$$

  then $\mathcal{C}_{\mathbb{T}}$ is a formal topology.

- If, furthermore, $\mathbb{T}$ *is a set*, then $\mathcal{C}_{\mathbb{T}}$ is an inductively generated formal topology.

Generalized geometric propositional theories, 2

In (Curi '10a) we proved that in particular the following classes of formal topologies cannot be proved to be inductively generated in CZF (+ REA + ...).

- (Non-trivial) Boolean formal topologies (see also (Gambino '06, Grayson '83));

- (Non-trivial) De Morgan (or extremally disconnected) formal topologies satisfying a weak decidability condition.

Thus, the generalized propositional geometric theories $\mathbb{T}$ corresponding to these formal topologies must have a proper class of axioms.

In (Caramello '08) the syntactic Booleanization and DeMorganization of a geometric theory are described.

The recalled results then imply that the sequents (axioms) one adds in e.g. the Booleanization of a propositional geometric theory $\mathbb{T}$, must form a proper class (in CZF) for every consistent $\mathbb{T}$.

(So unless one argues in an essentially impredicative way, the class of these axioms will not be finite, or denumerable).

Limits in **ImLoc**

Via minor modifications of constructions by Palmgren and Vickers for inductively generated formal spaces, one proves that **ImLoc** has all set-indexed limits.

However, while the constructions for formal spaces require REA, those for **ImLoc** can be derived in CZF, and even in CZF$_{exp}$. I.e., we have

- *in CZF$_{exp}$,* **ImLoc** *is a complete category*.

We discuss informally the construction of binary products, in particular of two *arbitrary* formal spaces.

Products

A formal space is usually presented in the form $X \equiv (S, \leq, \lhd)$, obtained by the one considered before by letting $a \lhd U \iff a \in A(U) \equiv I(\Phi_{\leq}, U)$.

Given formal spaces $X_1 \equiv (S_1, \leq_1, \lhd_1)$, $X_2 \equiv (S_2, \leq_2, \lhd_2)$, one would like to define

$$X_1 \times X_2$$

by letting $X_1 \times X_2 \equiv (S_1 \times S_2, \leq, \lhd)$, with $\lhd$ the least covering containing the pairs:

$$\Phi \equiv \{((a,b),\ U \times \{b\}) : a \lhd_1 U,\ b \in S_2\} \cup \{((a,b),\ \{a\} \times V) : a \in S_1,\ b \lhd_2 V\}$$

Note that Φ is proper class in general. In, e.g., IZF, Φ is a set and one may obtain $\lhd$ as the intersection of all cover relations containing Φ.

If S_1, S_2 are inductively generated by set-covering systems C_1, C_2, it suffices to consider the *set* of pairs

$$C \equiv \{((a,b), U \times \{b\}) : (a,U) \in C_1,\ b \in S_2\} \cup \{((a,b), a \times V) : a \in S_1,\ (b,V) \in C_2\}$$

C is a set, and gives rise to a covering system, so that, by the set inductive definition theorem, *assuming REA*, there is the least covering $\lhd$ containing C, defining the product of X_1, X_2, with obvious projections (CSSV '03).

Imaginary Products

On the other hand, any two (non-necessarily inductively generated) formal spaces X_1, X_2 are in particular imaginary locales (with, for $i = 1, 2$, $\Phi_i = \{(a, U) | a \lhd_i U \ \& \ U \subseteq \downarrow a\}$);

then their product $X_1 \times X_2$ must exist in **ImLoc**, *also without REA*, and is in fact given by $(S_1 \times S_2, \leq, \Phi')$, with Φ' the expected modification of Φ:

$\Phi' \equiv \{((a,b),\ U \times \{b\}) : (a \lhd_1 U \ \& \ U \subseteq \downarrow a),\ b \in S_2\} \cup \{((a,b),\ \{a\} \times V) | a \in S_1,\ (b \lhd_2 V \ \& \ V \subseteq \downarrow b)\}.$

For every $W \in \mathrm{Pow}(S_1 \times S_2)$ *the class* $A(W) \equiv I(\Phi'_{\leq}, W)$ exists, but is not a set in general;

so although we do have some information on the opens of this 'imaginary' product, precisely the information given by the (generators, and the) relations Φ', this information is not enough to describe a whole (class-) frame.

We take the class Φ' as our basic information: it says what should hold in the generated (class-)frame *if it existed*.

From that, *some* further concrete information can anyway be deduced, and a useful one, as will be shown below.

Examples

Examples of imaginary locales that may not be formal spaces are:

- The product of any two formal spaces is an imaginary locale, that need not be a formal space in CZF, with or without REA.
- The function space X^X for X locally compact is an imaginary locale that need not be a formal space in CZF without REA (Note: N^N is formal Baire space).
- More generally, if Φ is a covering system (i.e. Φ is in particular a set), without REA it will only define an imaginary locale.

Applications, 1

Basic applications of the concept of imaginary locales are of the following kind:

a locale X is *Hausdorff*, or T_2, iff the diagonal $X \to X \times X$ is closed.

Using imaginary locales this notion can be formulated for arbitrary (imaginary locales and) formal spaces, rather than just for inductively generated formal spaces.

One can then prove e.g. that

Every Boolean formal space is a Hausdorff formal space

even if Boolean formal spaces are not inductively generated.

More generally, in a topos, one has that every regular locale is Hausdorff.

Also, one may try to prove for general formal spaces standard results such as Vermeulen's characterization of compactness:

A locale X is compact iff the projection $\pi : X \times Z \to Z$ is closed for every Z.

Applications, 2

Imaginary locales allows for the development of point-free topology in weaker settings (as CZF *without* REA).

Johnstone-Tychonoff embedding theorem states that

Y is completely regular iff it can be embedded in the product $\prod_{h \in Hom(Y,[0,1])}[0,1]_h$.

$Hom(Y,[0,1])$ need not be a set in CZF, so that this product cannot be constructed in general.

However, using the (Strong) Collection scheme of CZF, one can select a subset I of $Hom(Y,[0,1])$ that suffices to carry out the proof:

Tychonoff embedding theorem (CZF$_{exp}$). *A formal space Y is completely regular iff there is a set $I \subseteq Hom(Y,[0,1])$ such that*

$$Y \hookrightarrow \prod_{h \in I}[0,1]_h.$$

Remark. Note that $\prod_{h \in I}[0,1]_h$ is only an imaginary locale in CZF and CZF$_{exp}$; it is a (inductively generated) formal space in CZF+REA.

Applications, 3

Imaginary locales can furthermore be used to obtain results concerning **FSp**, **IFSp**.

Using imaginary locales in a very natural way, one proves that

Theorem 3 *For every compact regular formal space* X_1*, and completely regular and i.g.* X_2*,*

$$Hom(X_1, X_2) \text{ is a set}^*$$

in CZF$_{exp}$*+AC*$_\omega$*, and in the choice-free system CZF*$_{exp}$*+*$(\forall a)SubColl(\mathbb{N}, a)$*.*

Corollary 1 **KCRegIFSp** *is locally small in CZF*$_{exp}$*+AC*$_\omega$ *and in CZF*$_{exp}$*+*$(\forall a)SubColl(\mathbb{N}, a)$*.*

In ZF+AC

$$\textbf{KCRegIFSp} \leftrightarrow \textbf{KHausSp}.$$

Corollary 2 *The class of points of a completely regular and i.g. formal topology is a set in CZF*$_{exp}$*+AC*$_\omega$*, and in CZF*$_{exp}$*+*$(\forall a)SubColl(\mathbb{N}, a)$*.*

Note that these results must fail in CZF$_{exp}$, by (Lubarsky & Rathjen '08).

$^*Hom(X_1, X_2)$ denotes the class of saturated continuous functions from X_1 to X_2.

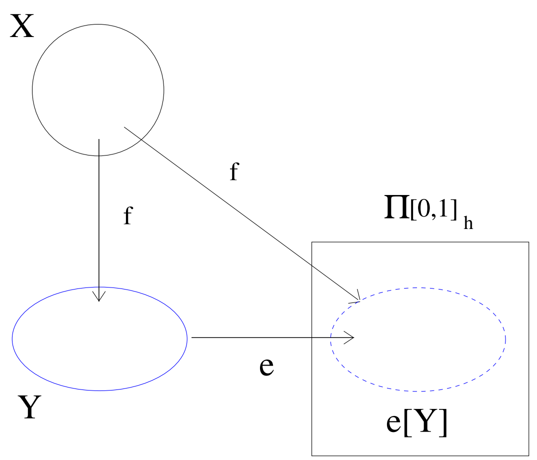
Idea of the proof

Given a compact regular space X, and an i.g. completely regular space Y, one proves that $Hom(X,Y)$ is a set by first showing that, in **ImLoc**,

$$Hom(X, \prod_{h\in I}[0,1]_h)$$

is a set, for every set I.

By the Johnstone-Tychonoff embedding theorem, one may then regard a completely regular space Y as a subspace of an imaginary locale $\prod_{h\in I}[0,1]_h$:



so we may think of the class of mappings $Hom(X,Y)$ as a subclass of $Hom(X, \prod_{h\in I}[0,1]_h)$.

Then, using the fact that Y is inductively generated, it is possible to separate a subset that indexes the class $Hom(X,Y)$.

Concluding remarks, open problems, possible developments

- Colimits in **ImLoc** (in particular coequalizers).
- The ‘structure’ associated with a frame presentation.
- A predicative version of Caramello’s DeMorganization of a locale.
- ‘Stability theorems’ using imaginary locales.
- Imaginary locales and Grothendieck sites; relation with Gambino’s version of Grothendieck site.