Independence results in constructive set theory and constructive type theory

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Structure of talk

- The first half consists of an overview of the submitted research project.
- The second half presents some of the independence results obtained so far (informally).

Constructive set theories and type theories

- Martin-Löf's *constructive type theories* CTT, and
- Aczel-Myhill's *constructive set theories* CST,

are the main formal systems for *constructive mathematics* (in the sense of Bishop).

CTT and CST are (generalized) *predicative*.

In particular, the collection Pow(X) of subsets of a set X is not a set in these systems.

Intuitionistic set theory and topos logic

Two other important intuitionistic systems are:

- H. Friedman's *Intuitionistic Set Theory*, IZF.

  A system for set theory with intuitionistic rather than classical logic, but 'as similar as possible' to ZF.

- *Topos logic* = *Higher-order Heyting Arithmetic*, HHA.

  The internal logic of elementary toposes.

IZF (resp. HHA) has *powersets* (resp. *power-sorts*) and *unrestricted separation* (resp. *full comprehension*).

So they are intuitionistic, but *impredicative*, systems.

General motivations, I

Albeit much weaker than HHA and IZF, CST and CTT have shown to permit the proof of high level results from

general topology;

functional analysis;

commutative algebra, algebraic geometry;

...

beside allowing for the formalization of Bishop's CM.

General motivations, II

It is then natural to try to understand the *limits* of these predicative systems.

Several independence results concerning CST, and in particular CZF (= Aczel's formulation of CST), have already been obtained

(Rathjen, Lubarsky, van den Berg & Moerdijk,...).

However, in almost every case, these results have established the independence of a certain set-theoretical axiom from the remaining axioms of the system.

The submitted project aimed at the *identification of independence results relative to standard fields as topology, algebra, algebraic geometry, etc.*

General motivations, III

Of course, independence results proved for IZF and/or HHA will often carry over to CST and CTT.

In particular the goal is then to identify results

> *provable in IZF, or in HHA* (i.e. 'intuitionistically valid'),
>
> *that cannot be proved in CST, CTT*.

In the second part of the talk, we shall discuss (very informally) two results of this kind in the area of (point-free) topology.

General motivations, IV

Use/meaning of these results:

- *Proof-theoretic*, reminiscent of Reverse Mathematics.
- *Applications in classical mathematics*:

  (certain subsystems and extensions of) CZF and CTT can be seen as *internal languages of categories*, as HHA w.r.t. toposes (cf. Simpson, Streicher, ...). Therefore,...
- *New notions*: refinements and improvements of known ones. Ex: the concept of *locale* (discussed below) w.r.t. the ordinary notion of *topological space*.
- *Philosophical*: at least for a particular kind of independence results, (discussed below), the results whose independence is proved can be regarded as *non-constructive* rather than just *underivable* in the given system.

Point-free topology

Over *classical* ZF, the following theorems *cannot* be proved:

- Tychonoff Theorem ($\Leftrightarrow AC$);
- Hahn-Banach Theorem ($\Leftarrow AC$);
- Gelfand Duality ($\Leftarrow AC$);
- Stone-Čech Compactification ($\Leftrightarrow PIT$);
- Gleason's Covering Theorem ($\Leftarrow PIT$);
- ...

All these (and many other) theorems become provable in *full generality* in (ZF and) IZF, *provided one replaces (the use of) topological spaces with (the use of) locales*.

Full generality: No restrictions (as instead in RM). Moreover, in most cases,

'point-free (i.e. localic) argument + AC $\Rightarrow$ usual version of T'.

Locales informally, I

Given a topological space $(X, \Omega(X))$,

$$(\Omega(X), \bigcup, \cap, \emptyset, X)$$

is a (bounded) complete lattice with $\bigvee \equiv \bigcup$, $\wedge \equiv \cap$.

$\Omega(X)$ also satisfies the *infinite distributive law*

$$U \wedge \bigvee_{i \in I} V_i = \bigvee_{i \in I} (U \wedge V_i),$$

for $U, V_i \in \Omega(X)$.

A *locale* (or frame) is a complete lattice satisfying the infinite distributive law.

Locales informally, II

With a suitable notion of *continuous function* locales form a category **Loc**.

Even classically, locales are ‘many more’ than those that can be obtained as $\Omega(X)$.

- E.g. every *non-atomic complete Boolean algebra*.

Locales have pleasant properties not enjoyed by topological spaces, so are of interest also in ZF+AC: J. Isbell, A. Joyal, A. Simpson, ...

Constructive aspects: Joyal, Banaschewski & Mulvey, Johnstone, Martin-Löf & Sambin, Coquand, Aczel, ...

Remark. L is a locale iff it is a complete Heyting algebra $\rightsquigarrow$ Heyting-valued models of constructive and intuitionistic set theories.

Point-free topology and constructive set theories, I

Locales allow then for proofs of the above mentioned classical theorems in IZF, HHA (that, as a consequence, can be interpreted in any topos).

Does the same hold for CST, CTT?

T. Coquand has shown that point-free versions of

- Tychonoff Theorem,
- Hahn-Banach Theorem,
- Vietoris construction,
- ...

can be obtained in CST, CTT (cf. **KGRPF-08**).

Point-free topology and constructive set theories, II

We will shaw, that, on the other hand, *by contrast with what happens in IZF, HHA*, the point-free versions of:

- Stone-Čech compactification in its full form,
- Gleason's covering theorem,

cannot be proved in CZF, CTT as well as in several, even impredicative, extensions of these systems.

Aczel's CZF, I

The constructive Zermelo-Fraenkel set theory CZF is Aczel's formulation of CST.

CZF is formulated in the same (first-order) language of ZF; it uses intuitionistic logic and has the following axioms and axiom schemes:

- Extensionality,
- Pairing,
- Union,
- Infinity,
- Set Induction: $\forall x[(\forall y \in x)\varphi(y) \rightarrow \varphi(x)] \rightarrow \forall x \varphi(x)$,
- Restricted Separation (i.e., Separation for bounded formulae),
- Strong Collection,
- Fullness.

Aczel's CZF, II

Strong Collection *For every set a, if $(\forall x \in a)(\exists y)\, \varphi(x, y)$, then there is a set b such that $(\forall x \in a)(\exists y \in b)\, \varphi(x, y)$ and $(\forall y \in b)(\exists x \in a)\, \varphi(x, y)$.*

Strong Collection $\Rightarrow$ Replacement, purely logically.

For sets a, b, let $mv(b^a)$ be the *class* of subsets r of $a \times b$ such that $(\forall x \in a)(\exists y \in b)\, (x, y) \in r$.

Fullness *Given sets a, b there is a subset c of the class $mv(b^a)$ such that for every $r \in mv(b^a)$ there is $r_0 \in c$ with $r_0 \subseteq r$.*

Myhill's Exponentiation Axiom in particular follows by Fullness.

Exponentiation *For sets a, b, the class of functions from a to b is a set.*

Aczel's CZF, III

One has

- IZF = CZF + Sep + Pow

then

$$\text{CZF} \subsetneq \text{IZF} \subsetneq \text{ZF}.$$

Moreover,

$$\text{CZF} + \text{EM} = \text{IZF} + \text{EM} = \text{ZF}.$$

Remark. Although Pow(X) is not a set in CZF, it is a *class*, the class of *subsets* of X.

Troelstra's Uniformity Principle

CZF is *consistent* with the following principles:

Troelstra's principle of uniformity

1. if $(\forall x)(\exists n \in \omega)\varphi(x, n)$, then $(\exists n \in \omega)(\forall x)\varphi(x, n)$.

Every set is subcountable

2. $(\forall x)(\exists U \in Pow(\omega))(\exists f) f : U \twoheadrightarrow x$

(van den Berg & Moerdijk, Lubarsky, Rathjen, Streicher, ...).

The Generalized Uniformity Principle

By 1, 2 one gets the *Generalized Uniformity Principle*:

GUP: For every set a, if $(\forall x)(\exists y \in a)\varphi(x, y)$, then $(\exists y \in a)(\forall x)\varphi(x, y)$.

with which CZF is then consistent. In fact, various extensions of CZF, including

$$\text{CZF} + (\text{REA} + \text{PA}) + \text{Sep}$$

are consistent with GUP.

Moreover, constructive type theory CTT is also consistent with a suitable formulation of this principle (Coquand & Petit).

CZF* (resp. CTT*) will denote any extension of CZF (resp. of CTT) that is compatible with GUP.

Locales in CZF

For X any set with *discrete* topology,

$$\Omega(X) = \mathsf{Pow}(X).$$

A locale in CZF is a partially ordered *class* L, that has $\bigvee, \wedge$, which satisfy the infinite distributive law, and that has a *set* B_L *of generators*:

every element of L is the join of a *subset* of B_L.

In IZF, the two notions are equivalent.

The Gleason cover of a compact regular space/locale, I

A space X is *extremally disconnected* iff

$$X = U^* \cup U^{**},$$

for all $U \in \Omega(X)$, where U^* is the largest open disjoint from U;

lattice-theoretically, U^* is given by $U \to \emptyset$.

The Gleason cover of a compact regular space (equivalently, of a compact Hausdorff space) is a pair

$$(\gamma X,\ e : \gamma X \twoheadrightarrow X),$$

with γX a *compact*, *regular* and *extremally disconnected* space, and e a continuous surjection that is minimal in a certain sense.

The Gleason cover of a compact regular space/locale, II

- Over ZF, γX exists for every compact regular space X *iff* PIT.
- In a topos, or in IZF, γL can be constructed for *every* compact regular *locale* L (Johnstone).

We shall see that, however, the existence of the Gleason cover of a compact regular locale can be *refuted* in CZF*+GUP, and is therefore not derivable in CZF* (and similarly for CTT*).

Stone's Lemma

A classical result of M. Stone states that:

the (compact) locale $Idl(B)$ of ideals of a Boolean algebra B is extremally disconnected if and only if B is complete.

This result also holds in topos logic, and gives the *intuitionistic* existence of compact extremally disconnected locales.

It is used in the construction of the Gleason cover of a locale L:

$$\gamma L \equiv Idl(L^{**}),$$

where L^{**} is the *Booleanization* of L.

Failure of Gleason, I

Recall that in CZF*, a locale is carried by a *proper class*.

One cannot therefore consider the class of ideals over L.

The fact that, for L a locale, $Idl(L)$ is not an admissible construction in CZF*, of course is not enough to conclude that the Gleason cover fails constructively to exist.

We have to show that an object with the properties characterizing the Gleason cover cannot exist.

Failure of Gleason, II

In contrast with what entailed by Stone's result we have:

Theorem. *No non-degenerate locale L can be proved to be extremally disconnected and compact in CZF*.*

Proof (Sketch). We show that in CZF* + GUP the assumption that L is extremally disconnected and compact is contradictory.

Let $\Omega \equiv \mathrm{Pow}(\{0\})$, and assume L is extremally disconnected and compact. Then,

$$\forall p \in \Omega,\ \top_L = (\bigvee\{\top_L : 0 \in p\})^* \ \vee\ (\bigvee\{\top_L : 0 \in p\})^{**}.$$

Using compactness and GUP, one shows that there is $\{x_1, ..., x_n\} \subseteq B_L$, with, *for all* $p \in \Omega$,

$$x_i \leq (\bigvee\{\top_L : 0 \in p\})^* \text{ or } x_i \leq (\bigvee\{\top_L : 0 \in p\})^{**},$$

for $i = 1, ..., n$, and such that $\top_L \leq x_1 \vee ... \vee x_n$.

Assuming $\neg(x_1 = \bot_L) \vee ... \vee \neg(x_n = \bot_L)$, one gets

$$(\forall p \in \Omega)(\neg(0 \in p) \text{ or } \neg\neg(0 \in p)),$$

i.e. [R]DML holds.

Failure of Gleason, III

As is easy to prove, [R]DML is inconsistent with GUP, so that

$$\neg(\neg(x_1 = \bot_L) \vee ... \vee \neg(x_n = \bot_L)).$$

This gives

$$\neg\neg(x_1 = \bot_L \ \&...\& \ x_n = \bot_L).$$

On the other hand, since L is non-degenerate, from $\top_L \leq x_1 \vee ... \vee x_n$ one gets

$$\neg(x_1 = \bot_L \ \&...\& \ x_n = \bot_L),$$

so that L is not compact and extremally disconnected in CZF*+GUP. ■

We have therefore the following strong refutation of the existence of Gleason covers.

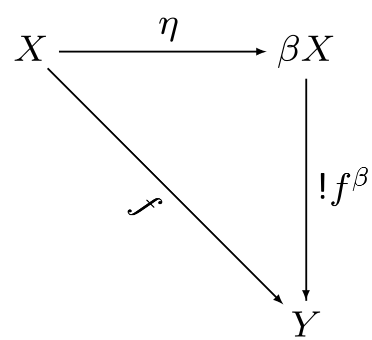
Corollary. *The Gleason cover of no (non-degenerate) compact regular locale can be defined in CZF* (and similarly for CTT*).*

Existence of Stone-Čech compactification, I

The Stone-Čech compactification of a space or locale X is its compact completely regular reflection, i.e., it is a continuous map

$$\eta : X \to \beta X,$$

with βX *compact* and *completely regular*, which satisfies the following universal property:



for all compact completely regular Y, and all continuous $f : X \to Y$.

Remark. This universal property gives a bijection $Hom(X,Y) \cong Hom(\beta X, Y)$, for every compact completely regular Y.

Existence of Stone-Čech compactification, II

As said, this compactification exists in a topos, and in IZF, *for every locale* L.

In CZF+REA we have

Theorem. *The Stone-Čech compactification* βX *of a locale* X *exists if, and only if,* $Hom(X, [0,1])$ *is a set.*

In particular, one has

Theorem. *For every locally compact* X, βX *exists in CZF+sREA+DC.*

Existence of Stone-Čech compactification, III

Thus, in contrast with the Gleason cover, Stone-Čech compactification *does constructively exist* for a class of locales. However:

Theorem. *The Stone-Čech compactification of a non-degenerate Boolean locale X cannot be defined in CZF*.*

To prove the theorem we need the following results:

Lemma. *A bijection exists between the class of elements of any boolean locale X and $Hom(X, \mathit{Pow}(\{0, 1\}))$.*

and

Theorem. *The full subcategory of compact regular locales* **KRLoc** *is locally small in CZF.*

Remark. With PIT, **KRLoc** $\leftrightarrow$ **KHausSp**.

Existence of Stone-Čech compactification, IV

Proof of the theorem. Assume X is a Boolean locale.

If βX existed, by the Theorem, $Hom(\beta X, \mathsf{Pow}(\{0,1\}))$ would be a set in CZF.

Moreover, by the universal property of β,

$$Hom(X, \mathsf{Pow}(\{0,1\})) \cong Hom(\beta X, \mathsf{Pow}(\{0,1\})).$$

Thus $Hom(X, \mathsf{Pow}(\{0,1\}))$ would be a set too. By the Lemma, X would then be a set in CZF*. However, using the consistency of CZF* with GUP, one may prove that no non-degenerate locale can be proved to have a set of elements in CZF*.

Thus, βX cannot exist in CZF*. ■

Corollary. For X Boolean, $Hom(X, \mathcal{R})$, $Hom(X, [0,1])$ *are proper classes in CZF*.*

A possible necessary condition for constructivity

A possible *necessary* condition for an argument to be defined constructive is that it may be formulated within an extension of CZF or CTT that is compatible with GUP (i.e., in CZF* or CTT*).

The systems IZF, HHA are not among these extensions.

Remark. Far from being a *sufficient* condition:

$$CZF + Sep \, (\cong 2PA)$$

is compatible with GUP. So the given one is a *very liberal* criterion.

For the constructivist who accepts the given condition for constructivity, the independence results recalled above can be read as saying that Stone-Čech compactification of a Boolean locale, or the existence of Gleason covers, *although topos-valid, are non-constructive theorems.*